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LETTERS TO THE EDITOR

TRANSVERSE VIBRATIONS OF CIRCULAR, ANNULAR PLATES WITH SEVERAL COMBINATIONS OF BOUNDARY CONDITIONS

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1. INTRODUCTION

A recent publication deals with the exact determination of lower natural frequency coefficients of circular, annular plates with a free inner boundary while the outer contour is either clamped or simply supported [1]. The study was motivated by design needs of transducer elements. It was pointed out by the authors in that study, that certain Rayleigh–Ritz type determinations provided numerical results which were in gross disagreement with eigenvalues previously available in the open literature and considered as "exact". It was concluded that the latter did not possess enough accuracy.

More recently the authors of reference [1] were confronted with the need of knowledge of the lower natural frequencies of natural, circular plates with the four combinations of boundary conditions, see Figure 1: clamped at both boundaries (Case I); clamped at $r = a$ and simply supported at $r = b$ (Case II); simply supported at $r = a$ and clamped at $r = b$ (Case III); simply supported at both edges (Case IV).

2. EXACT MATHEMATICAL SOLUTION OF VIBRATING THIN, CIRCULAR ANNULAR PLATES

The exact mathematical solution of the problem under study is well known (see, for instance, Leissa's classical treatise [2]). Normal modes of transversal vibration are described by the eigenfunction

$$W_n(r, \theta) = [A_n \mathrm{J}_n(kr) + B_n \mathrm{Y}_n(kr) + C_n \mathrm{I}_n(kr) + D_n \mathrm{K}_n(kr)]\, \mathrm{e}^{in\theta}, \quad (1)$$

where J_n and Y_n are the Bessel functions of the first and second kind, respectively, and I_n and K_n are modified Bessel functions of the first and second kind, respectively. The parameter k is given by

$$k^4 = (\rho h/D)\omega^2 \quad (2)$$

where ρ is the density of the plate material, h is the plate thickness, D is the flexural rigidity and ω the circular natural frequency of the structural system.

Consider now the case where the edge $r = a$ is clamped. The governing boundary conditions are

$$W_n(a, \theta) = (\partial W_n/\partial r)(a, \theta) = 0. \quad (3)$$

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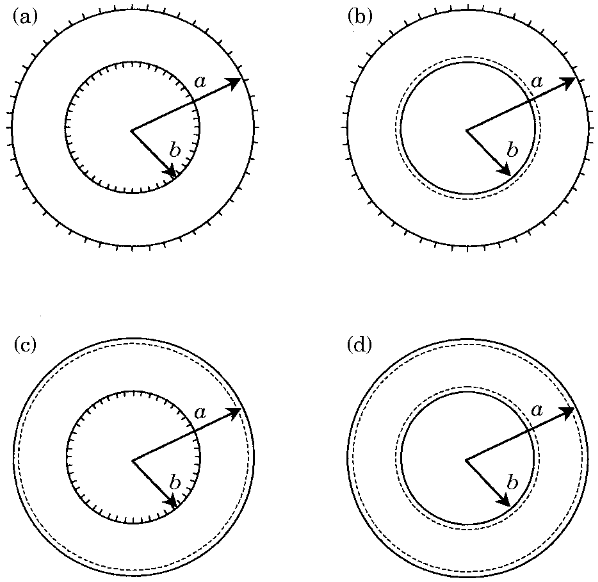

Figure 1. Different mechanical configurations executing transverse vibrations considered in the present study: (a) Case I; (b) Case II; (c) Case III; (d) Case IV.

If the contour $r = a$ is simply supported one has

$$W_n(a, \theta) = 0, \qquad -D\left[\frac{\partial^2 W_n}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial W_n}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W_n}{\partial \theta^2}\right)\right]\Bigg|_{r=a} = 0 \tag{4, 5}$$

In order to calculate derivatives of Bessel functions one makes use of well known recurrence relations [2]. In the case of $n = 0$ (axisymmetric modes) they are quite simple

$$(\mathrm{d}/\mathrm{d}r)\mathrm{J}_0(kr) = -kJ_1(kr), \qquad (\mathrm{d}/\mathrm{d}r)\mathrm{Y}_0(kr) = -kY_1(kr),$$
$$(\mathrm{d}/\mathrm{d}r)\mathrm{I}_0(kr) = kI_1(kr), \qquad (\mathrm{d}/\mathrm{d}r)\mathrm{K}_0(kr) = -kK_1(kr). \tag{6}$$

In order to satisfy the condition (5) for $n = 0$ one proceeds in the following straightforward manner. The Bessel function J_0 and Y_0 satisfy the differential equation

$$\mathrm{d}^2 W_1/\mathrm{d}r^2 + (1/r)\,\mathrm{d}W_1/\mathrm{d}r + k^2 W_1 = 0. \tag{7}$$

Accordingly,

$$\mathrm{d}^2 W_1/\mathrm{d}r^2 = -(1/r)\,\mathrm{d}W_1/\mathrm{d}r - k^2 W_1. \tag{8}$$

Adding the term $((\nu/r)\,\mathrm{d}W_1/\mathrm{d}r)$ to both sides one obtains

$$\mathrm{d}^2 W_1/\mathrm{d}r^2 + (\nu/r)\,\mathrm{d}W_1/\mathrm{d}r = -k^2 W_1 + (1/r)\,\mathrm{d}W_1/\mathrm{d}r(\nu - 1). \tag{9}$$

which expresses the corresponding component of the boundary condition at $r = a$ in terms of the original Bessel functions J_0 and Y_0, and their first order derivatives.

On the other hand, the modified Bessel functions satisfy

$$\mathrm{d}^2 W_2/\mathrm{d}r^2 + (1/r)\,\mathrm{d}W_2/\mathrm{d}r - k^2 W_2 = 0. \tag{10}$$

Following a similar procedure to the one previously explained one obtains

$$\mathrm{d}^2 W_2/\mathrm{d}r^2 + (\nu/r)\,\mathrm{d}W_2/\mathrm{d}r = k^2 W_2 + (1/r)\,\mathrm{d}W_2/\mathrm{d}r(\nu - 1). \tag{11}$$

TABLE 1

Frequency coefficients $\Omega_{0n} = \sqrt{\rho h/D}\,\omega_{0n} a^2$ for circular annular plates clamped at both boundaries (Case I)

b/a	n = 0	1	2
0·1	27·2805	28·9157	36·6172
0·2	34·6092	36·1032	41·8196
0·3	45·3462	46·6435	51·1388
0·4	61·8722	62·9959	66·6716
0·5	89·2500	90·2302	93·3212
0·6	139·6190	140·4795	143·1338
0·7	248·4021	249·1638	251·4805
0·8	559·1625	559·8416	561·8899
0·9	2237·1762	2237·7855	2239·6157

Note: the results are independent of ν

The null displacement conditions at each boundary and relations (6) have been used for the case of a clamped edge. When the boundary is simply supported one makes use of the zero displacement condition and implements relations (10) and (11). The resulting frequency determinants for the four cases depicted in Figure 1 are shown in the Appendix A (equations (A1)–(A4)).

3. NUMERICAL RESULTS

The eigenvalues have been computed for $\nu = 0{\cdot}3$ and 1/3 (obviously in the case where both boundaries are clamped, the eigenvalues are independent of Poisson's ratio, see equation (A-1)). Use has been made of Maple V [3]. Tables 1–4 depict values of $\Omega_{0n} = \sqrt{\rho h/D}\,\omega_{0n} a^2$ for $b/a = 0{\cdot}1, 0{\cdot}2, \ldots, 0{\cdot}9$ for the four structural configurations shown in Figure 1. The subscript "0" denotes the fact that only the first eigenvalue corresponding to each value of n has been calculated. Clearly, Ω_{00} is the fundamental frequency coefficient for each situation. Comparing the results of Table 1 with those contained in table 2.17

TABLE 2

Frequency coefficients $\Omega_{0n} = \sqrt{\rho h/D}\,\omega_{0n} a^2$ for circular, annular plates clamped at $r = a$ and simply supported at $r = b$ (Case II)

	$\nu = 0{\cdot}3$			$\nu = 1/3$		
	n			n		
b/a	0	1	2	0	1	2
0·1	22·7014	25·2826	35·4062	22·5846	25·2090	35·3919
0·2	26·7371	29·2487	37·6201	26·6196	29·1576	37·5785
0·3	33·7654	35·9055	42·7314	33·6526	35·8072	42·6661
0·4	45·0444	46·8386	52·4360	44·9323	46·7345	52·3526
0·5	63·9732	65·4855	70·1359	63·8562	65·3731	70·0360
0·6	98·9228	100·2105	104·1305	98·6937	100·0839	104·0112
0·7	174·4077	175·5159	178·8659	174·2535	175·3631	178·7172
0·8	389·7206	390·6838	393·5825	389·5112	390·4750	393·3758
0·9	1549·7372	1550·5819	1553·1177	1549·3544	1550·1993	1552·7360

TABLE 3

Frequency coefficients $\Omega_{0n} = \sqrt{\rho h/D}\omega_{0n}a^2$ for circular, annular plates simply supported at $r = a$ and clamped at $r = b$ (Case III)

	$\nu = 0{\cdot}3$			$\nu = 1/3$		
	n			n		
b/a	0	1	2	0	1	2
0·1	17·7893	19·3941	26·7169	17·8376	19·4401	26·7578
0·2	22·7144	24·2722	30·0880	22·7674	24·3229	30·1332
0·3	29·9777	31·4025	36·2434	30·0367	31·4595	36·2952
0·4	41·1932	42·4827	46·6382	41·2605	42·5483	46·6990
0·5	59·8199	60·9870	64·6313	59·8987	61·0644	64·7048
0·6	94·1674	95·2267	98·4727	94·2636	95·3219	98·5648
0·7	168·5240	169·4892	172·4139	168·6494	169·6139	172·5363
0·8	381·4534	382·3362	384·9948	381·6373	382·5197	385·1768
0·9	1534·1369	1534·9475	1537·3810	1534·4969	1535·3072	1537·7400

of reference [2]† one notices a reasonably good agreement for values of $b/a \leqslant 0{\cdot}5$. The agreement is considerably better when comparing with the results contained in Table 2.18 of reference [2].‡

The agreement is, again, quite reasonable between the frequency coefficients contained in Table 2 and those presented in Table 2.20 of reference [2] which were determined by Vogel and Skinner [5] for $\nu = 0{\cdot}3$. As shown in Table 2 the values of Ω_{0n} decrease slightly as ν increases from 0·3 to 1/3.

For the configuration defined as (III) in Figure 1 there is good agreement between the results shown in Table 3 and those contained in Table 2.24 of references [2, 5] for $\nu = 0{\cdot}3$. From the analysis of Table 3 one concludes that the eigenvalues increase as ν varies from 0·3 to 1/3. There is also good coincidence with the results depicted in Table 2.23 of references [2, 6].

In the case where both boundaries are simply supported the largest discrepancy with the eigenvalue Ω_{00} obtained by Raju (Table 2.25 of references [2, 6]) is for $b/a = 0{\cdot}2$ and $\nu = 1/3$; see Table 4. It is observed that Ω_{0n} decreases as ν increases from 0·3 to 1/3.

† Obtained from reference [4]; ‡ Obtained from reference [5].

TABLE 4

Frequency coefficients $\Omega_{0n} = \sqrt{\rho h/D}\omega_{0n}a^2$ for circular, annular plates simply supported at both boundaries (Case IV)

	$\nu = 0{\cdot}3$			$\nu = 1/3$		
	n			n		
b/a	0	1	2	0	1	2
0.1	14.4847	16.7759	25.9357	14.4386	17.7635	25.9662
0.2	16.7796	19.2222	27.2404	16.7298	19.1960	27.2530
0.3	21.0791	23.3171	30.2734	21.0357	23.2871	30.2705
0.4	28.1225	30.1092	36.1560	28.0853	30.0794	36.1431
0.5	40.0431	41.7973	47.0887	40.0111	41.7691	47.0702
0.6	62.1542	63.7065	68.3717	62.1264	63.6805	68.3508
0.7	110.0634	111.4433	115.5851	110.0389	111.4197	115.5638
0.8	247.0904	248.3236	252.0234	247.0687	248.3022	252.0029
0.9	987.2715	988.3793	991.7026	982.2521	988.3599	991.6834

4. CONCLUSIONS

It is hoped that the eigenvalues presented in this paper will be useful to acousticians and mechanical designers since considerable effort has been placed in providing very good accuracy. On the other hand considerable credit must be given to previous researchers who, with very modest computational tools, provided extremely useful numerical results.

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APPENDIX A:

(*see equations A1–A4 overpage*)

APPENDIX A: FREQUENCY EQUATIONS FOR CASES I TO IV

$$\begin{vmatrix} J_0(ka) & Y_0(ka) & I_0(ka) & K_0(ka) \\ J_0(kb) & Y_0(kb) & I_0(kb) & K_0(kb) \\ -J_1(ka) & -Y_1(ka) & I_1(ka) & -K_1(ka) \\ -J_1(kb) & -Y_1(kb) & I_1(kb) & -K_1(kb) \end{vmatrix} = 0 \qquad \text{(A1)}$$

$$\begin{vmatrix} J_0(ka) & Y_0(ka) & I_0(ka) & K_0(ka) \\ J_0(kb) & Y_0(kb) & I_0(kb) & K_0(kb) \\ -J_1(ka) & -Y_1(ka) & I_1(ka) & -K_1(ka) \\ (-kb)J_0(kb)-(v-1)J_1(kb) & (-kb)Y_0(kb)-(v-1)Y_1(kb) & (kb)I_0(kb)+(v-1)I_1(kb) & (kb)K_0(kb)-(v-1)K_1(kb) \end{vmatrix} = 0 \qquad \text{(A2)}$$

$$\begin{vmatrix} J_0(ka) & Y_0(ka) & I_0(ka) & K_0(ka) \\ J_0(kb) & Y_0(kb) & I_0(kb) & K_0(kb) \\ (-ka)J_0(ka)-(v-1)J_1(ka) & (-ka)Y_0(ka)-(v-1)Y_1(ka) & (ka)I_0(ka)+(v-1)I_1(ka) & (ka)K_0(ka)-(v-1)K_1(ka) \\ -J_1(kb) & -Y_1(kb) & I_1(kb) & -K_1(kb) \end{vmatrix} = 0 \qquad \text{(A3)}$$

$$\begin{vmatrix} J_0(ka) & Y_0(ka) & I_0(ka) & K_0(ka) \\ J_0(kb) & Y_0(kb) & I_0(kb) & K_0(kb) \\ (-ka)J_0(ka)-(v-1)J_1(ka) & (-ka)Y_0(ka)-(v-1)Y_1(ka) & (ka)I_0(ka)+(v-1)I_1(ka) & (ka)K_0(ka)-(v-1)K_1(ka) \\ (-kb)J_0(kb)-(v-1)J_1(kb) & (-kb)Y_0(kb)-(v-1)Y_1(kb) & (kb)I_0(kb)+(v-1)I_1(kb) & (kb)K_0(kb)-(v-1)K_1(kb) \end{vmatrix} = 0 \qquad \text{(A4)}$$